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ON THE COMPUTED PRESSURES FOR NAVIER–STOKES PROBLEMS AT INCREASING REYNOLDS NUMBERS USING THE PENALTY FINITE ELEMENT METHOD

R. KRISHNAN AND G. F. CAREY

Aerospace Engineering and Engineering Mechanics, University of Texas at Austin, Austin, TX 78712, U.S.A.

DISCUSSION

The reduced integration penalty finite element method has been frequently used to compute approximate velocity solutions to Navier–Stokes problems.[1–4] The velocity approximation can be post-processed to obtain an approximate pressure solution. Numerical studies for Stokes flow[5,6] have shown that for problems where the data (boundary conditions or the forcing function) are 'rough', these computed pressures may exhibit oscillations for certain elements and reduced integration schemes, whereas smoother pressures are obtained for other ('stable') elements. An example of such a problem is the familiar 'driven cavity' problem. For this problem, both in the case of Stokes flow and low Reynolds number Navier–Stokes flow, when the 4-node bilinear element with 1-point Gauss integration of the penalty term is used, the pressures computed from the velocity field are found to be oscillatory, whereas when the 9-node biquadratic element with 1-point integration of the penalty term is used, smooth pressure profiles are obtained.

Recently, we have been investigating the use of continuation techniques and iterative methods for computing approximate solutions at higher Reynolds numbers. On examining the computed pressures for the driven cavity we observed that as the Reynolds number increased, the local pressure oscillations for the bilinear element diminish and by $Re = 2000$ the pressure profiles at representative sections appear smooth.

As an example, results are now given for the cavity problem and bilinear 1-point element for calculations on a uniform mesh of size $h = 1/32$ and with penalty parameter $\varepsilon = 10^{-4}$. The pressure profile at $Re = 10$ along the section $y = 17/64$ is given in Figure 1 (marked $+$) and is seen to contain local pressure oscillations. The projected pressure obtained by averaging the four element pressures adjacent to a node is also given and is smooth (marked $\times$). Analogous results are also observed for Stokes flow ($Re = 0$). However, the computed pressure profile at $y = 17/64$ for $Re = 2000$ appears smooth (Figure 2).[†] Similar behaviour is observed at other representative sections and is not shown here. Results with penalty parameter $\varepsilon = 10^{-5}$ and $\varepsilon = 10^{-3}$ were essentially the same.

Approximate solutions were also computed using the mixed method with C^0 biquadratic velocity and C^0 bilinear pressure on uniform meshes with $h = 1/16$ and $h = 1/25$. This element is known to be stable. The pressure solution is smooth as anticipated and qualitatively agrees with that obtained with the penalty method. However, quantitatively the pressure profiles agree

[†] P. Gresho and his colleagues at Lawrence Livermore Laboratory have repeated the computations and verified our observation that the pressures at higher *Re* are smoother than those at low *Re*.

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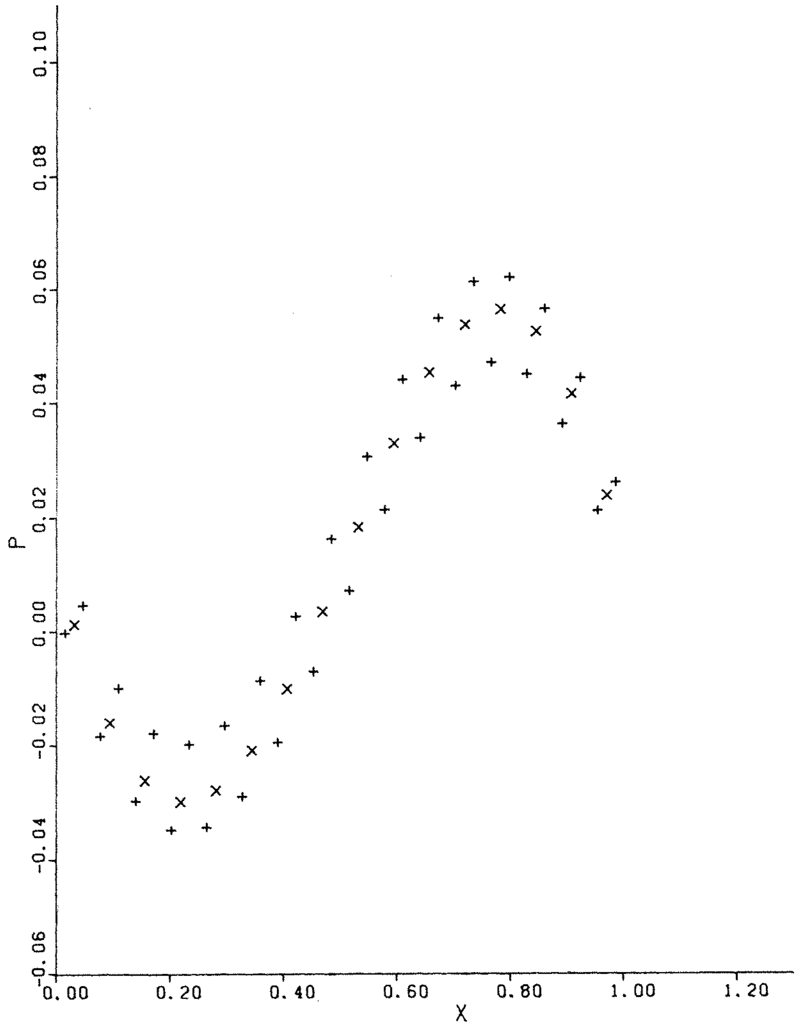

Figure 1. Computed pressures (+) $p_h^\varepsilon(x, \frac{17}{64})$ and averaged pressures ($\times$) for $Re = 10$ (uniform mesh, $h = 1/32$ with penalty parameter $\varepsilon = 10^{-4}$)

well at low *Re* but were not as close at higher *Re*, this difference possibly being due to the inadequacy of the mesh for flows at higher Reynolds numbers.

CONCLUDING REMARKS

1. These calculations indicate that, for the frequently studied cavity problem, local oscillations in the computed pressures diminish as *Re* increases. Although one cannot generalize from this particular case and set of results, they do raise interesting questions regarding the method and use of elements for problems where non-linear effects are significant.

2. In an attempt to interpret the above result we considered the projection of the computed pressure into the orthogonal basis functions for the pressure space described by Johnson and

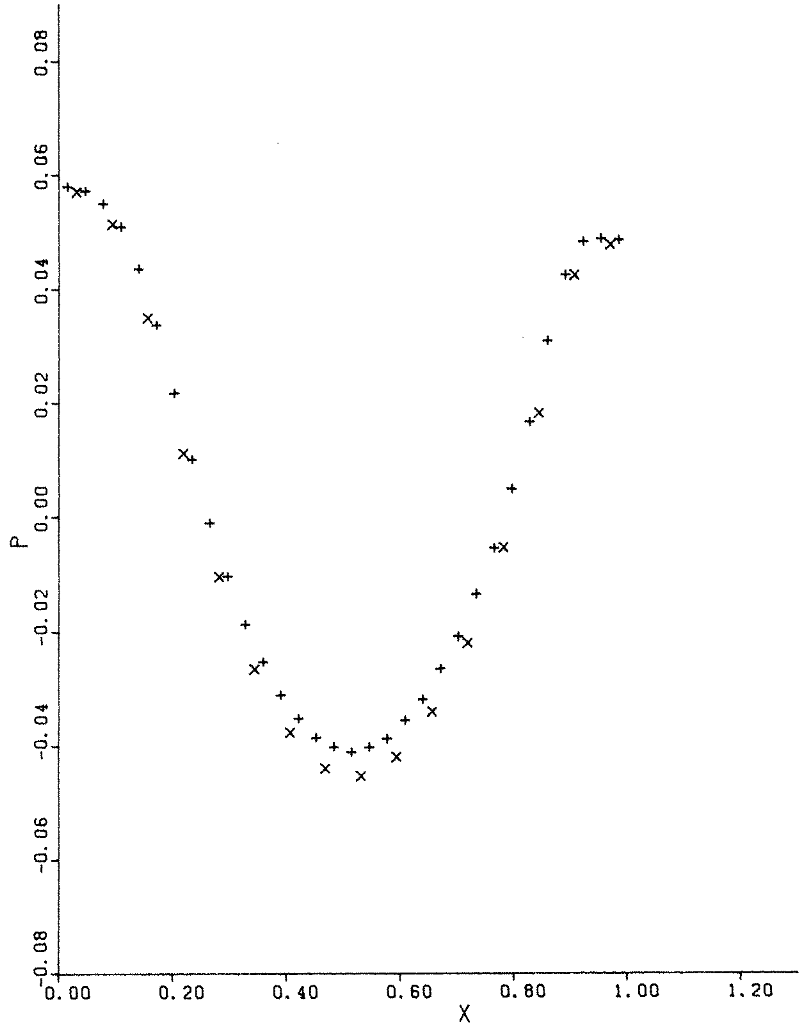

Figure 2. Computed pressures (+) $p_h^\varepsilon(x, \frac{17}{64})$ and averaged pressures (×) for $Re = 2000$ (uniform mesh, $h = 1/32$ with penalty parameter $\varepsilon = 10^{-4}$)

Pitkäranta.[7,8] No conclusive behaviour or result regarding the smoothness of the pressures at $Re = 2000$ could be obtained from examining the coefficients in this projection.

3. We also computed the components C_1 and C_2 of the projection in the constant and 'chequerboard' functions. The values of C_1 and C_2 are computed from p_h^ε simply by taking the L^2 inner product with the constant and chequerboard bases. We obtain $C_1 = (3.66)10^{-10}$, $(4.47)10^{-12}$ and $C_2 = (2.39)10^{-10}$, $(2.94)10^{-12}$ for $Re = 10$ and 2000, respectively. These values are small as one would expect from theory—they are the components of p_h^ε in ker B_h^* (where B_h^* is the discrete gradient operator) and we know that $p_h^\varepsilon \in (\ker B_h^*)^\perp$ for this formulation.

One point we wish to emphasize here is that the oscillation (even for Stokes flow $Re = 0$) is not associated with the chequerboard and constant bases. The oscillations arise from other bases in the pressure space.

4. The fact that the properties of the chequerboard mode are used in constructing a smooth filtered pressure approximation is independent of the above point: Johnson and Pitkäranta[7] give a rigorous explanation of the filtering scheme and its convergence.

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